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A SURVEY AND COMPARISON OF METHODS FOR DETERMINING CONFIDENCE BOUNDS ON SYSTEM RELIABILITY FROM SUB-SYSTEM DATA

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A SURVEY AND COMPARISON OF METHODS FOR DETERMINING CONFIDENCE BOUNDS ON SYSTEM RELIABILITY FROM SUBSYSTEM DATA*

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In this paper, methods of obtaining lower confidence bounds on the reliability of non-maintained systems are surveyed, and numerical comparisons are given. It is assumed that failure data have been collected from life tests performed on prototypes of the various subsystems which make up the system, but that the system has not been tested as a whole, for some particular reason. The reason may be, for example, expense or simply that it is virtually impossible to test the entire system without destroying it. Methods applicable to series and/or parallel and more logically complex systems are discussed.

OPTIMUM CONFIDENCE BOUNDS FOR SPECIFIC SYSTEM MODELS

There are two types of models for which one can theoretically calculate, from subsystem failure data, confidence bounds on system reliability which are known to be optimum in some sense. For a series system made up of k independent subsystems each having exponentially distributed failure time,T, there exists a lower confidence bound which is most accurate (has the highest probability of being close to the true system reliability) for all values of system reliability among exact bounds which are unbiased. (A family of lower confidence bounds $\underline{\theta}$ (X) on θ at confidence level $1-\alpha$ is unbiased if Prob $[\underline{\theta}(X)\leq\theta']\leq 1-\alpha$ for all $\theta'<\theta$ and for all values of nuisance parameters. The restriction of unbiasedness is necessary here because of the nuisance parameters $\lambda_1, \ldots, \lambda_k$, the failure rates for the k independent subsystems.) For this model, series system reliability $R(t_m)$ at time $t_m>0$ is equal to $\prod_{j=1}^{k} \exp(-t_m\lambda_j) = \exp(-t_m\sum_{j=1}^{k}\lambda_j)$, $\lambda_j>0$, $j=1, \ldots, k$, so that one can also think of obtaining an upper confidence bound on system failure rate, $\phi=\sum_{j=1}^{k}\lambda_j$.

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The optimum lower confidence bound on series-system reliability, derived for two subsystems with exponential failure data by Lentner and Buehler (1963) and generalized to $k \geq 2$ subsystems by El Mawaziny (1965), depends upon the assumption that for the *j*th subsystem, n_j prototypes have been tested until r_j, with $1 \leq r_j \leq n_j$, failures occur, $j=1, \ldots, k$. One observes for the *j*th subsystem, $t_{i,j}$, $i=1, \ldots, r_j$, and computes $z_j = \sum_{i=1}^{r_j} t_{i,j} + (n_j - r_j) t_{r,j}$, $j=1, \ldots, k$. Calculation of a confidence bound by El Mawaziny's method must be performed iteratively on a computer, and if the product of the number of subsystems and the total number of failures is large, problems of loss of precision result. See Mann (1970).

The other system model for which a method of obtaining optimum confidence bounds on system reliability has been derived is one in which binomially distributed attribute or pass-fail data are collected. No assumptions are made concerning the form of failure-time distributions; the assumptions are that the system is serial or parallel-redundant, the subsystems and the subsystem prototype tests are independent, the failure distribution for any specified subsystem is the same for each prototype of that subsystem tested and the duration of each test is the intended operating time applying to the particular subsystem in the system. For the binomially distributed data obtained for this model, Buehler (1957) defines a small-sample method of obtaining confidence bounds on system reliability. These bounds, however, like binomial confidence bounds for a single component, are conservative in general rather than exact because of the discreteness of the numbers of failures. In order to obtain a confidence bound which is exact in general and uniformly most accurate unbiased, a random number uniformly distributed on (0,1) must be generated and used in calculating the bound. (Confidence bounds which depend upon a uniform random variate in addition to the failure data will be referred to as randomized, while the conservative bounds which do not depend upon a generated random number will be called nonrandomized.)

Lipow and Riley (1959) have used Buehler's definition to compute and tabulate nonrandomized lower confidence bounds on series system reliability R_s for systems containing one, two, or three subsystems and attribute subsystem data. In all of their tabulations, the number n of items tested is the same for each component because of the problem of ordering the failure combinations in calculating each confidence bound.

The value of n ranges from 5 to several hundred. The number of failures is not required to be the same for each component and ranges from zero to about $n/2$. The tabulated lower confidence bounds on series system reliability are known to be optimum (correspond to shortest intervals with plus one as an upper bound) among nonrandomized confidence bounds based on the subsystem attribute data for two independent subsystems when the sum of the numbers of failures is less than $2\sqrt{n}$. For three subsystems, the problem of ordering the failure combinations prevents the authors from claiming optimality for the bounds based on their particular ordering of the failure combinations. Steck (1957) gives in graphical form the same sort of conservative nonrandomized confidence bounds for parallel-system reliability R_p, for systems composed of two independent subsystems and at most one observed failure.

APPROXIMATE AND NON-OPTIMAL EXACT CONFIDENCE BOUNDS FOR THE EXPONENTIAL MODEL

For each of these system models, there are a number of methods that have been derived for approximating the optimum bounds. In some cases, the approximations were derived before the optimum bounds. In other cases, they were derived because of the problem involved with calculating the optimum bounds.

For the exponential series-system model, the method of Kraemer (1963) yields bounds which are exact and depend upon only the smallest sample mean subsystem failure time. This method gives confidence bounds which are very inaccurate (i.e., they exhibit a high probability of being far from the true system reliability) unless one failure only occurs for either every subsystem or every subsystem except one. In these two special cases, Kraemer's method gives the optimum confidence bounds under certain conditions.

The methods of Sarkar (1971) and Lieberman and Ross (1971), developed for the exponential series-system model, depend upon combinations of failure times selected from the various subsystems rather than the sufficient statistics, $Z_1, \ldots, Z_k$, observed as $z_1, \ldots, z_k$. As Lieberman and Ross (1971) point out, in using their method one can obtain different

bounds by ordering the subsystem tests differently, and thus combining the subsystem test data differently. The confidence intervals of Lieberman and Ross are exact and stochastically shorter in general than those of both Sarkar and Kraemer.

Grubbs (1969) uses what will be called here the <u>fiducial</u> approach to approximate the distribution of series-system failure rate, given the exponential subsystem failure data. That is, he assumes that since the distribution of $2z_j$ is $\lambda_j^{-1}\chi^2(2r_j)$, then λ_j is distributed as $\chi^2(2r_j)/2z_j$, $j=1, \ldots, k$. Grubbs employs an approach which uses the mean $\sum_{j=1}^{k} r_j/z_j$ and variance $\sum_{j=1}^{k} r_j/z_j^2$ of the fiducial distribution of system failure rate $\phi=\sum_{j=1}^{k}\lambda_j$ to fit a noncentral Chi-square distribution.

Burnett and Wales (1961) and Levy and Moore (1967) suggest for this series-system model, and also for other models which are more logically complex than a series system, Monte Carlo simulation of the fiducial distribution of system reliability, given the subsystem failure data. Such an approach yields, for the series-system model essentially the same result as the method of Grubbs; see Mann (1970) and Grubbs (1971). The fiducial approach is equivalent to obtaining Bayesian bounds on system reliability at time t_m with the prior density for the <u>j</u>th subsystem failure rate $\lambda_j>0$ equal to λ_j^{-1}, $j=1, \ldots, k$.

Although lower confidence bounds on system reliability obtained by the fiducial method are optimum for one subsystem and approach the optimum confidence bounds as the numbers of failures increase for all subsystems, they are conservative in general. That is, for this series-system model, the fiducial lower bound on system reliability is always less than the corresponding optimum exact classical lower confidence bound, and sometimes by a considerable amount. This can be seen in Table 1.

The confidence bounds on series-system reliability derived by El Mawaziny and Buehler (1967) for the exponential model depend upon the asymptotic normality of system failure rate, given the subsystem data as it is assumed given in the derivation of the optimum bounds of El Mawaziny. That is, one assumes z_1, and $u_2=z_1-z_2, \ldots, u_k=z_1-z_k$, are given, where the subscript 1 is arbitrarily assigned. As shown in Table 1, a lower

confidence bound obtained by this method is always greater than the corresponding optimum bound when the number of failures is small for any subsystem. One of the most serious problems inherent in this method of obtaining lower confidence bounds on system reliability is that subsystems for which a single failure occurs are ignored. The consequences of this fact are exhibited in Table 1.

The methods of Rosenblatt (1963) and Madansky (1965) yield asymptotic lower confidence bounds on series-system reliability which, like the method of El Mawaziny and Buehler, tend to be very much too large when only a single failure occurs for any subsystem. These methods are discussed in more detail in the section dealing with confidence bounds based on binomial subsystem data.

The Approximately Optimum confidence bounds of Mann and Grubbs (1972) are based on a combination of the approach of Grubbs (1971) and that of El Mawaziny and Buehler (1967) for the exponential series-system model. One conditions on z_1, and $u_2, \ldots, u_k$, as in deriving the optimum confidence bounds on R_s and determines expressions for the approximate conditional mean and variance of system failure rate ϕ. The conditional variate Z_1, given $\underset{\sim}{u}$, belongs to the Koopman-Darmois exponential family with parameter ϕ and nuisance parameters $\lambda_1, \ldots, \lambda_k$, so that by the discussion given by Lehmann (1959, sec. 4.3), a uniformly most accurate unbiased upper confidence bound for ϕ can be obtained from z_1 and $\underset{\sim}{u}$.

Once the approximate conditional mean and variance of system failure rate are calculated, one can approximate very accurately the noncentral Chi-square conditional distribution of system failure rate by another central Chi-square distribution [see Patnaik (1949)] and thus obtain approximately optimum lower confidence bounds on series-system reliability. A Wilson-Hilferty (1931) approximation of Chi-square by a Gaussian variate can be used to facilitate the calculations when noninteger degrees of freedom are encountered. The expressions for the conditional mean m and variance v of system failure rate ϕ derived by Mann and Grubbs have been simplified by Mann (1972) and are

$$m = \sum_{j=1}^{k} (r_j - 1)/z_j + z_{(1)}^{-1} \tag{1}$$

TABLE 1. LOWER 50%, 75%, AND 90% CONFIDENCE BOUNDS, ON SERIES-SYSTEM RELIABILITY: EXPONENTIAL SUBSYSTEM FAILURE DATA, WITH MISSION TIME $t_m = 1$

r_j = Number of Failures	Observed Values of $z_j = r_j\hat{\theta}_j$					Confidence Level	El Mawaziny Optimum Lower Bound	Approx. χ^2 (fiducial)	Approx. χ^2 Based on (1),(2) & (3)	El Mawaziny-Buehler Normal Approx.
	z_1	z_2	z_3	z_4	z_5					
4,2,2,3,3	9.919	13.996	26.897	26.311	62.439	0.75	0.485	0.406	0.479	0.525
4,2,2,3,3	28.412	13.992	9.990	98.228	44.667	0.50	0.670	0.569	0.657	0.710
1,2,2,3,3	0.363	4.609	21.045	25.328	11.419	0.90	0.010	0.006	0.009	0.426
1,2,2,3,3	9.976	2.381	20.275	10.044	31.118	0.75	0.242	0.173	0.233	0.365
1,2,2,3,3	0.363	4.609	21.045	25.328	11.419	0.50	0.179	0.100	0.156	0.595
2,3,4	23.059	32.504	37.188	--	--	0.50	0.812	0.757	0.804	0.831
2,3,4	73.758	16.084	57.503	--	--	0.90	0.671	0.647	0.669	0.733
2,3,4	73.758	16.084	57.503	--	--	0.50	0.795	0.766	0.791	0.827
2,3,4	19.753	31.249	21.321	--	--	0.75	0.692	0.629	0.688	0.721
2,2,2	16.966	42.258	15.318	--	--	0.50	0.828	0.761	0.824	0.863
2,2,2	16.966	42.258	15.318	--	--	0.75	0.764	0.691	0.760	0.81[illegible]
4,3,2	42.733	45.791	31.890	--	--	0.90	0.772	0.725	0.766	0.801
4,3,2	84.794	36.989	4.163	--	--	0.90	0.359	0.344	0.358	0.526
4,3,2	28.274	44.690	9.480	--	--	0.90	0.566	0.524	0.559	0.658
4,3,2	171.581	29.062	11.293	--	--	0.90	0.649	0.614	0.642	0.737
4,3,2	78.668	54.968	57.296	--	--	0.90	0.849	0.815	0.846	0.868
4,3,2	23.527	56.469	23.476	--	--	0.75	0.750	0.700	0.746	0.777

and

$$v = \sum_{j=1}^{k} (r_j - 1)/z_j^2 + z_{(1)}^{-2} \tag{2}$$

where $z_{(1)}$ is the smallest of the z's. To approximate the optimum lower bound on $R_s(t_m)$ at confidence level $1-\alpha$ using the Wilson-Hilferty transformation, one calculates

$$\underline{R}_s(t_m) = \exp\left[-t_m m \left\{1-v/(9m^2) + Z_{1-\alpha} v^{1/2}/(3m)\right\}^3\right], \tag{3}$$

where Z_γ is the $\underline{100\gamma}$th percentile of a standard normal distribution. Confidence bounds based on formulas (1), (2) and (3) are compared in Table 1 with the optimum classical exact confidence bounds of El Mawaziny and with the fiducial bounds and the asymptotic approximation of El Mawaziny and Buehler. The data used for the comparisons were generated in conjunction with the study described by Mann (1970). The Approximately Optimum lower confidence bounds $\underline{R}_s(t_m)$ on $R_s(t_m)$ agree to within about a unit in the second decimal place with optimum lower confidence bounds of El Mawaziny (1965).

APPROXIMATE AND NON-OPTIMAL EXACT CONFIDENCE BOUNDS FOR THE BINOMIAL MODEL

For series or parallel-redundant models and the case in which only pass-fail binomially distributed data are collected for each subsystem, many methods involving large- or small-sample approximations or Bayesian techniques have been derived for obtaining nonrandomized confidence bounds on the probability of successful operation of a system. Among the large-sample methods is one derived by Madansky (1965). Extension of Madansky's method to logically complex systems has been accomplished by Myhre and Saunders (1968b). In using this method, one parameterizes so as to determine iteratively parameter values which make the negative of the logarithm of the likelihood ratio equal to one half of a specified percentile of the Chi-square distribution with one degree of freedom; see Wilks (1938). In determining the likelihood ratio by this method, one must maximize the joint binomial density function subject to a constraint specified by the equation relating system reliability to the subsystem reliabilities.

The principal disadvantage associated with use of the likelihood-ratio procedure is that any component exhibiting zero failures in a life-test situation is ignored in the determination of a series-system reliability confidence bound and tends to bias parallel system confidence bounds on the high side. Thus, the procedure yields lower confidence bounds that are too high whenever zero failures are observed during the life test, no matter how large the sample size for all components. This method is, in fact, not appropriate for highly reliable systems.

Another method suggested for obtaining confidence bounds on system reliability is one based on the asymptotic normality of the unbiased <u>simulation</u> estimator (often equivalent to the maximum-likelihood estimator) of system reliability. Such a procedure and its theoretical rationale are discussed at length by Rosenblatt (1963). Easterling (1972) investigated a variation of this procedure in which psuedo numbers of system tests and successes are determined from the estimate of the variance of the maximum-likelihood estimator of system reliability. These numbers are then substituted into the incomplete Beta function, and confidence bounds obtained as in determining usual optimum nonrandomized confidence bounds for a single component and binomial sampling. Easterling found his approximation to be better for several series, parallel and series-parallel systems which he investigated than that based on the asymptotic normality of maximum-likelihood estimators and comparable to that based on the asymptotic Chi-square distribution of the logarithm of the likelihood-ratio function (which is more difficult to implement).

Woods and Borsting (1968) have approximated the distribution of $-2 \ln \hat{R}_s$, with $\hat{R}_s$ the maximum-likelihood estimator of series-system reliability, by a continuity-corrected Gamma distribution. The approximation appears to give exact bounds when sample sizes are large, but does not appear to perform well for sample sizes as small as 10.

Murchland and Weber (1972) describe a method similar to that detailed by Rosenblatt for obtaining confidence bounds on the reliability of a complex system. These authors, however, suggest the use of Chebyshev inequalities rather than settling on any particular assumptions concerning

the distribution (asymptotic or otherwise) of the unbiased simulation estimator of system reliability, and their bounds involve the expression for the exact rather than the asymptotic variance of the estimator of R_s. Their procedure presents no particular computational difficulties as long as the system remains logically simple. Calculation of their variance estimate for the reliability estimator becomes very complicated, however, as the system increases in complexity.

A more serious difficulty inherent in this method and all methods dependent upon the maximum-likelihood or simulation estimator of system reliability is similar to that given for the likelihood-ratio method. It is the fact that any component exhibiting no failures during the life tests is ignored in the determination of bounds for strictly serial systems and yields a lower confidence bound on parallel-system reliability that is always unity. The maximum-likelihood and simulation estimators are also inappropriate for providing the basis for confidence bounds on the reliability of highly reliable systems.

Table 2 gives for series systems the optimum nonrandomized confidence bounds of Lipow and Riley (1959), likelihood-ratio (LR) and maximum-likelihood (ML) approximations calculated by Myhre and Saunders (1968a) and Easterling's modified maximum likelihood (MMLI) approximations, all for data sets in which at least one failure occurs for each component. Results based on the Woods and Borsting approximation are shown in Table 3.

Small-sample approximate confidence bounds on R_s for an independent series system and binomial data have been derived by Garner and Vail (1961), Connor and Wells (1962), Abraham (1962), and Lindstrom and Madden [see Lloyd and Lipow (1962)]. The first two of these approaches use various methods of combining confidence bounds on subsystem reliability to obtain the desired bounds on system reliability. The other two use binomial or Poisson approximations for certain statistics. Some of these methods are sensitive to inequality of sample sizes for subsystems. Lower confidence bounds obtained by most of these approximate methods have been compared by the use of three sets of data by Schick and Prior (1966) with optimum nonrandomized bounds obtained using results of Lipow

and Riley (1959). The data apply to series systems composed of two subsystems, and in each of the three cases the sample sizes are equal. Only the Lindstrom and Madden method compares favorably with the Lipow and Riley (1959) bounds. The Lindstrom and Madden method, however, is one which is sensitive to unequal sample sizes for subsystems.

One can also obtain Bayesian bounds on system reliability by choosing, for a system composed of k components, a prior density function for the <u>j</u>th component reliability, R_j, j=1, ..., k, and then simulating (or obtaining through a Mellin-transform technique) the posterior distribution of system reliability, given the component failure data. Zimmer, Prairie and Breipohl (1965) and Springer and Thompson (1966) have suggested the use of prior densities on component reliabilities that are uniform on (0,1). Mastran (1968) and Parker (1972) prefer to assign prior densities in such a way that the prior density for <u>system</u> reliability is uniform.

Results of Raiffa and Schlaifer (1961) suggest that any prior density approximating the prior yielding an optimum confidence bound should be <u>improper</u> (i.e., be such that its integral cannot be made equal to unity by affixing a constant factor) as is the prior density function $p(R) = R^{-1}$ (or, equivalently, $p(-\ell nR)$ uniform on the positive half real line) associated with the optimum nonrandomized binomial confidence bound for a single component. Results of Fertig (1972) and Mann (1970) relating to series systems with exponential subsystem data corroborate this conjecture. Their results show further that for the exponential model, the prior densities on component reliabilities (which we shall refer to henceforth as <u>posterior</u> prior densities) associated with the optimum classical lower confidence bound on series-system reliability contain component failure data. The Approximately Optimum lower confidence bounds on R_s based on the conditional mean m and variance v of system failure rate, given by formulas (1) and (2), implicitly depend upon an assumption of a posterior prior density for λ_j of $\lambda_j^{-2+z_j/(kz_{(1)})}$, j=1, ..., k, where $z_{(1)}$ is the smallest of the z's. Finally, results of Mann (1971) indicate that both randomized and nonrandomized optimum confidence bounds for the reliability of series and parallel systems with

binomial subsystem data are associated with prior densities containing failure data.

Use of component-reliability posterior prior density functions which are not approximately optimum can yield Monte Carlo lower confidence bounds having a high probability of being inordinately far from the true system reliability. For example, if one uses as a prior for R_j, $p(R_j) = R_j^{-1}$, j=1, ..., k, with k>1, the optimum (nonrandomized) prior density for k=1, then the lower confidence bound ained (referred to henceforth as <u>fiducial</u> bounds) for system reliability R are uniformly lower than the optimum nonrandomized lower confidence bounds. For a parallel system the difference between the two types of confidence bounds is not large unless sample sizes n_j, j=1, ..., k are very small. However, one obtains for a series system with $n_1=10$, $n_2=9$, $n_3=9$, $x_1=2$, $x_2=1$ and $x_3=0$, where the x's are numbers of failures, an optimum nonrandomized lower confidence bound of 0.504 and a fiducial bound of 0.348. These results are shown by Mann (1971) and other comparisons with optimum nonrandomized confidence bounds are shown in Tables 3 and 4.

Using a bound based on uniform prior densities for the component reliabilities is equivalent to using the fiducial bound with n_j, j=1, ..., k, each increased by one. Thus, for the data set above, the bound based on uniform priors for component reliabilities is 0.387. For more discussion concerning the use of uniform prior densities, see Mann and Fertig (1972).

Harris (1971) has derived a procedure for obtaining nonrandomized and optimum randomized confidence bounds on products and quotients of Poisson parameters. Since for large sample sizes and small probabilities the Poisson distribution can be used to approximate binomial distributions, Harris has suggested using his randomized results to approximate <u>parallel</u> system reliability confidence bounds under appropriate conditions. The appropriate conditions are, of course, large sample sizes and high reliabilities for all components.

Harris compares numerically, for a selection of Poisson failure data applying to parallel systems composed of two subsystems, his Poisson lower bounds and the Poisson approximation of Buehler (1957) to

the optimum nonrandomized lower confidence bounds on R_p. Harris' randomized lower confidence bounds on R_p are optimum (uniformly most accurate unbiased) for Poisson data, but his nonrandomized lower confidence bounds are extremely conservative.

The method of Harris is rather difficult to implement, involving the calculation of the percentiles of the modified Bessel function; and, unfortunately, there is no procedure by which Harris' bounds can be made to apply to at least moderately reliable series systems with binomial subsystem data.

Mann (1971) has used the approach used by Mann and Grubbs (1972) for exponential-subsystem failure data to approximate both optimum randomized and optimum nonrandomized lower confidence bounds on series- and parallel-system reliability for systems with binomial subsystem data. For sample sizes large and the numbers of failures small, Mann's Approximately Optimum (AO) randomized lower confidence bounds on parallel-system reliability R_p agree very well with the optimum randomized Poisson bounds of Harris (see Table 5). To obtain the Approximately Optimum randomized lower confidence bounds on R_p, one first calculates

$$m_p(\delta) = \sum_{j=1}^{k} \sum_{i=x_{j+1}}^{n_j} (1/i) + .5/x_{(1)} - .5 \sum_{j=1}^{k} \delta/x_j \qquad (4)$$

and

$$v_p(\delta) = \sum_{j=1}^{k} \sum_{i=x_j+1}^{n_j} (1/i)^2 + .5/x_{(1)} - .5 \sum_{j=1}^{k} \delta/x_j \qquad (5)$$

with $x_{(1)}$ equal to the smallest of the x's and δ equal to both +1 and -1. Then $m_p' = u\, m_p(1) + (1-u)\, m_p(-1)$ and $v_p' = u\, v_p(1) + (1-u)\, v_p(-1)$ where u is a realization of a random variate U uniform on (0,1). Use of m_p' and v_p' with the Wilson-Hilferty (1931) transformation for the approximate noncentral Chi-square distribution of $-\ell n(1-R_p)$ gives

$$\text{Prob}\left\{1-R_p \le \exp\left[-m_p' \left\{1-v_p'/(3\, m_p')^2 + Z_\alpha \sqrt{v_p'}/(3\, m_p')\right\}^3\right]\right\} \cong 1-\alpha\ , \qquad (6)$$

with Z_γ the <u>100γ</u>th percentile of a standard normal distribution.

One obtains a similar approximation to optimum exact lower confidence bounds on R_s by calculating $m_s(\delta)$ and $v_s(\delta)$ [$m_p(\delta)$ and $v_p(\delta)$, respectively, with x_j replaced by n_j-x_j and $x_{(1)}$ replaced by $n_{(1)}-x_{(1)}$, the smallest value of n_j-x_j, $j=1, \ldots, k$]. One then substitutes in (6), R_s for $1-R_p$, $\geq$ for $\leq$, $Z_{1-\alpha}$ for Z_α, $m_s'=u\ m_s(-1)+(1-u)m_s(1)$ for m_p' and $v_s'=u\ v_s(-1)+(1-u)v_s(1)$ for v_p' to obtain the AO randomized approximately exact lower confidence bound on R_s. The expressions (4) and 5) given for $m_p(\delta)$ and $v_p(\delta)$ are approximations for differences of polygamma functions, derivatives of the logarithm of the Gamma function; see Mann and Fertig (1972).

Expressions for m_p and m_s and v_p and v_s from which Approximately Optimum nonrandomized lower confidence bounds on system reliability can be obtained are similar to, though somewhat more complicated than, those given by formulas (4) and (5), respectively. Table 2 gives, for a series system with failures occurring in all subsystems, comparisons of these confidence bounds with the optimum nonrandomized confidence bounds calculated by Lipow and Riley (1959), those based on the likelihood-ratio (LR) and maximum-likelihood (ML) approximations, calculated by Myhre and Saunders (1968a), and that based on the modified maximum-likelihood (MMLI) approximation of Easterling (1971). Table 3 gives comparisons of the optimum nonrandomized lower confidence bounds, the AO nonrandomized confidence bounds, fiducial bounds and those based on the Gamma approximation of Woods and Borsting (1968) to the distribution of $-\ell n\hat{R}_s$, with $\hat{R}_s$ the maximum-likelihood estimator of series-system reliability. Table 4 applies to parallel systems and compares optimum nonrandomized and AO nonrandomized lower confidence bounds with fiducial bounds on R_p.

Table 5 applies to parallel systems, large sample sizes and small numbers of failures, and exhibits the optimum nonrandomized Poisson lower confidence bounds on R_p of Buehler (1957), the optimum "randomized" Poisson bounds, calculated by Harris (1971) from his formulas by using the expected value, .5, of a random variate uniform on (0,1), and the nonrandomized version of Harris' lower confidence bounds on R_p, the bottom of the range of possible lower confidence bounds to be obtained by randomizing with his method. Also shown in Table 5 are the AO lower

TABLE 2. NONRANDOMIZED LOWER CONFIDENCE BOUNDS ON RELIABILITY
FOR A SERIES SYSTEM OF TWO OR THREE COMPONENTS

	No. of Failures			Confidence Level									
				.90					.95				
k = 2	x_1		x_2	ML	LR	Opti-mum	AO	MMLI	ML	LR	Opti-mum	AO	MMLI
n = 10	1		1	.655	.629	.607	.615	.585	.611	.571	.548	.560	.530
	1		2	.545	.529	.497	.512	.489	.495	.473	.443	.459	.436
	2		2	.456	.451	.445	.443	.441	.405	.397	.392	.392	.391
	1		4	.347	.350	.344	.344	.318	.292	.301	.298	.298	.271
	2		3	.373	.375	.364	.371	.362	.320	.326	.304	.325	.315
n = 20	1,		2	.756	.739	.716	.720	.709	.728	.700	.677	.682	.671
	2,		2	.701	.687	.683	.683	.669	.670	.647	.643	.645	.631
	1,		3	.697	.683	.660	.665	.655	.665	.643	.620	.626	.616
	2,		3	.647	.636	.622	.624	.619	.614	.597	.582	.586	.580
	3,		3	.599	.591	.585	.582	.570	.565	.551	.544	.543	.532
k = 3	x_1	x_2	x_2										
n = 20	1,	1,	1	.760	.743	.747	.740	.721	.732	.705	.709	.703	.684
	1,	1,	2	.704	.690	.693	.683	.669	.673	.651	.644	.645	.631
	1,	2,	2	.654	.643	.639	.637	.619	.621	.604	.598	.599	.580
	1,	2,	3	.605	.596	.595	.591	.587	.571	.557	.544	.553	.549
n = 30	1,	2,	3	.723	.714	.705	.707	.669	.698	.683	.674	.677	.638
	1,	1,	1	.835	.822	.825	.818	.803	.816	.794	.796	.791	.775
	2,	2,	2	.725	.715	.712	.711	.703	.700	.685	.681	.681	.672
n = 50	1	2	4	.805	.798	.789	.791	.788	.788	.776	.767	.770	.766
	1	1	2	.874	.865	.861	.859	.852	.860	.845	.841	.840	.833
n = 100	1,	1,	2	.936	.931	.929	.927	.923	.929	.920	.918	.916	.313
	2,	3,	5	.866	.861	.858	.858	.856	.855	.848	.844	.845	.842

TABLE 3. NONRANDOMIZED NINETY PERCENT LOWER CONFIDENCE BOUNDS ON SERIES SYSTEM RELIABILITY FOR UNEQUAL SAMPLE SIZES

	No. of Tests			No. of Failures			Optimum	AO	Fiducial	Gamma
k = 1	n_1			x_1						
	7			1			.547	.548	.548	.257
	7			0			.720	.722	.722	1.00
	10			4			.354	.354	.354	.383
	10			1			.663	.664	.664	.386
k = 2	n_1	n_2		x_1	x_2					
	15	10		0	0		.794	.795	.678	1.00
	5	10		0	1		.587	.592	.475	.386
k = 3	n_1	n_2	n_3	x_1	x_2	x_3				
	10	5	4	1	1	0	.403	.416	.225	.096
	10	10	9	0	1	1	.588	.582	.417	.450
	10	9	9	2	1	0	.504	.505	.348	.180

TABLE 4. NONRANDOMIZED NINETY PERCENT LOWER CONFIDENCE BOUNDS ON PARALLEL SYSTEM RELIABILITY

	No. of Tests			No. of Failures			Optimum	AO	Fiducial
k = 2	n_1	n_2		x_1	x_2				
	10	10		0	1		.968	.968	.956
	2	2		0	1		.54	.537	.492
	10	10		0	0		.988	.981	.976
	50	50		0	1		.9985	.9983	.9978
	20	20		0	1		.991	.990	.987
k = 3	n_1	n_2	n_3	x_1	x_2	x_3			
	5	5	5	0	0	0	.997	.990	.987
	10	10	10	0	0	0	.999	.998	.998

TABLE 5. NINETY PERCENT LOWER CONFIDENCE BOUNDS ON PARALLEL SYSTEM RELIABILITY, LARGE SAMPLE SIZES

Number of Failures			Buehler's Poisson Approximation	Harris' Nonrandomized Confidence Bound (Poisson Approx.)	Harris' "Randomized" Confidence Bound	AO Nonrandomized Conf. Bound ($m_1=m_2=m_3=100$)	AO "Randomized" Conf. Bound ($m_1=m_2=m_3=100$)
x_1	x_2						
3	3		$1-41.2/(n_1\ n_2)$	$1-48.6/(n_1\ n_2)$	$1-41.6/(n_1\ n_2)$	$1-42.0/n^2$	$1-41.7/n^2$
1	4		$1-18.8/(n_1\ n_2)$	$1-23.3/(n_1\ n_2)$	$1-18.4/(n_1\ n_2)$	$1-20.0/n^2$	$1-19.4/n^2$
2	2		$1-16.8/(n_1\ n_2)$	$1-21.1/(n_1\ n_2)$	$1-17.0/(n_1\ n_2)$	$1-16.9/n^2$	$1-16.4/n^2$
10	0		$1-23.4/(n_1\ n_2)$	$1-33.9/(n_1\ n_2)$	$1-24.5/(n_1\ n_2)$	$1-24.2/n^2$	$1-22.7/n^2$
x_1	x_2	x_3					
1	2	1	- - - - - - -	$1-40.0/(n_1\ n_2\ n_3)$	$1-27.0/(n_1\ n_2\ n_3)$	$1-20.7/n^3$	$1-27.2/n^3$
2	3	3	- - - - - - -	$1-186/(n_1\ n_2\ n_3)$	$1-143/(n_1\ n_2\ n_3)$	$1-133/n^3$	$1-146/n^3$

confidence bounds of Mann (1971) corresponding to the optimum nonrandomized lower confidence bounds on R_p and the optimum "randomized" lower confidence bounds (calculated by setting the uniform random variate U equal to .5 in evaluating m'_p and v'_p.

METHODS APPLYING TO COMPLEX SYSTEMS

For binomial subsystem data, several of the methods described above for application in particular to series and parallel systems, have been used to obtain confidence bounds on the reliability of logically complex coherent systems. Coherent systems are defined here as defined by Birnbaum, Esary, and Saunders (1961); i.e., systems which if they perform when a given set of components perform also perform when a set of components containing the given set perform and which fail when all their components fail and perform when all components perform.

Use has been made of the approximate methods of Rosenblatt (1963), Madansky (1965), as generalized by Myhre and Saunders (1968b), Easterling (1971) and Murchland and Weber (1972), discussed earlier. The modified maximum likelihood (MMLI) method of Easterling probably gives as good an approximation to the optimum nonrandomized confidence bounds as any of these methods (see Table 2), and is not difficult to implement. As noted earlier, however, none of the methods based on the likelihood ratio or the simulation or maximum-likelihood estimator $\hat{R}_s$ of system reliability should be used if the system is highly reliable so that zero failures are exhibited for any subsystem. The results in Table 3 applying to the Gamma approximation to the distribution of $-\ell n\hat{R}_s$ of Woods and Borsting (1968) demonstrate how zero failures are ignored in the application of these methods for obtaining lower confidence bounds on series-system reliability, and Harris (1971) demonstrates similar results for the maximum-likelihood and likelihood-ratio methods applied to parallel systems.

Monte Carlo simulation procedures have been used widely, though incorrectly, for obtaining confidence bounds on the reliability of a logically complex coherent system. Because of the fact that subsystem failure data appear in the posterior prior density functions associated

with optimum confidence bounds for series and parallel systems, a straightforward Monte Carlo approach for logically complex systems seems infeasible if meaningful confidence bounds are to be obtained.

A new approach is given by Mann and Fertig (1972). It is assumed that for a logically complex coherent system, all of the most basic subsystems which are made up of effective components only (earlier called subsystems, but tested as a single unit) have been identified. It is assumed further that for each of these basic subsystems, either m_s and v_s or m_p and v_p applying to AO nonrandomized confidence bounds have been determined, accordingly as each subsystem is series or parallel. (One can also randomize using this procedure and thus calculate various values of m_s', v_s', m_p' and v_p'.)

The next step is to consider each subsystem for which means and variances have been calculated as a single effective component with an effective number n^* of prototypes subjected to life test and exhibiting an effective number x^* of failures, with n^* and x^* not necessarily integers. Each x^* and n^* must be determined iteratively. Then, having found values of n^* and x^* for each effective component applying to all most basic subsystems, these new data are combined with each other and with the failure data from other single components to determine effective component sample sizes and failure numbers for the next most basic set of subsystems in the total system. This procedure is continued until the effective failure number and sample size are determined for the entire system. The details involved in the process are given by Mann and Fertig. (1972).

This method is rather difficult to implement if the system is extremely complex, but provides an approach for obtaining confidence bounds that are approximately optimum for any coherent system, no matter how reliable.

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